

Exotic \mathbb{R}^4 and quantum field theory

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Abstract. Recent work on exotic smooth \mathbb{R}^4 's, i.e. topological \mathbb{R}^4 with exotic differential structure, shows the connection of 4-exotics with the codimension-1 foliations of S^3 , $SU(2)$ WZW models and twisted K-theory $K_H(S^3)$, $H \in H^3(S^3, \mathbb{Z})$. These results made it possible to explicate some physical effects of exotic 4-smoothness. Here we present a relation between exotic smooth \mathbb{R}^4 and operator algebras. The correspondence uses the leaf space of the codimension-1 foliation of S^3 inducing a von Neumann algebra $W(S^3)$ as description. This algebra is a type III_1 factor lying at the heart of any observable algebra of QFT. By using the relation to factor II , we showed that the algebra $W(S^3)$ can be interpreted as Drinfeld-Turaev deformation quantization of the space of flat $SL(2, \mathbb{C})$ connections (or holonomies). Thus, we obtain a natural relation to quantum field theory. Finally we discuss the appearance of concrete action functionals for fermions or gauge fields and its connection to quantum-field-theoretical models like the Tree QFT of Rivasseau.

1. Introduction

The construction of quantum theories from classical theories, known as quantization, has a long and difficult history. It starts with the discovery of quantum mechanics in 1925 and the formalization of the quantization procedure by Dirac and von Neumann. The construction of a quantum theory from a given classical one is highly non-trivial and non-unique. But except for few examples, it is the only way which will be gone today. From a physical point of view, the world surround us is the result of an underlying quantum theory of its constituent parts. So, one would expect that we must understand the transition from the quantum to the classical world. But we had developed and tested successfully the classical theories like mechanics or electrodynamics. Therefore one tried to construct the quantum versions out of classical theories. In this paper we will go the other way to obtain a quantum field theory by geometrical methods and to show its equivalence to a quantization of a classical Poisson algebra.

The main technical tool will be the noncommutative geometry developed by Connes [1]. Then intractable space like the leaf space of a foliation can be described by noncommutative algebras. From the physical point of view, we have now an interpretation of noncommutative algebras (used in quantum theory) in a geometrical context. So, we need only an idea for the suitable geometric structure. For that purpose one formally considers the path integral over spacetime geometries. In the evaluation of this integral, one has to include the possibility of different smoothness structures for spacetime [2, 3]. Brans [4, 5, 6] was the first who considered exotic

smoothness also on open smooth 4-manifolds as a possibility for space-time. He conjectured that exotic smoothness induces an additional gravitational field (*Brans conjecture*). The conjecture was established by Asselmeyer [7] in the compact case and by Ślădkowski [8] in the non-compact case. Ślădkowski [9, 10, 11] discussed the influence of differential structures on the algebra $C(M)$ of functions over the manifold M with methods known as non-commutative geometry. Especially in [10, 11] he stated a remarkable connection between the spectra of differential operators and differential structures. But there is a big problem which prevents progress in the understanding of exotic smoothness especially for the \mathbb{R}^4 : there is no known explicit coordinate representation. As the result no exotic smooth function on any such \mathbb{R}^4 is known even though there exist families of infinite continuum many different non diffeomorphic smooth \mathbb{R}^4 . This is also a strong limitation for the applicability to physics of non-standard open 4-smoothness. Bizaca [12] was able to construct an infinite coordinate patch by using Casson handles. But it still seems hopeless to extract physical information from that approach.

This situation is not satisfactory but we found a possible solution. The solution is a careful analysis of the small exotic \mathbb{R}^4 by using foliation theory (see especially [13, 14]) to derive a relation between exotic smoothness and codimension-1 foliations (see Theorem 3). By using noncommutative geometry, this approach is able to produce a von Neumann algebra via the leaf space of the foliation which can be interpreted as the observable algebra of some QFT (see [15]). Fortunately, our approach to exotic smoothness is strongly connected with a codimension-1 foliation of type *III*, i.e. the leaf space is a factor *III*₁ von Neumann algebra. Especially this algebra is the preferred algebra in the local algebra approach to QFT [15, 16]. Recently, this factor *III* case was also discussed in connection with quantum gravity (via the spectral triple of Connes) [17].

In this paper we present a deep connection between an exotic \mathbb{R}^4 as variation of the usual Minkowski space with \mathbb{R}^4 -topology. We start with some introductory material in the next section. In section 3 we present the main result about a relation between exotic \mathbb{R}^4 and type *III*₁ factor von Neumann algebras leading to a possible relation to Quantum field theory (QFT). This relation is presented in the last section with some remarks about the construction of the action functional.

2. From exotic \mathbb{R}^4 to operators algebras

Einstein's insight that gravity is the manifestation of geometry leads to a new view on the structure of spacetime. From the mathematical point of view, spacetime is a smooth 4-manifold endowed with a (smooth) metric as basic variable for general relativity. Later on, the existence question for Lorentz structure and causality problems (see Hawking and Ellis [18]) gave further restrictions on the 4-manifold: causality implies non-compactness, Lorentz structure needs a codimension-1 foliation. Usually, one starts with a globally foliated, non-compact 4-manifold $\Sigma \times \mathbb{R}$ fulfilling all restrictions where Σ is a smooth 3-manifold representing the spatial part. But other non-compact 4-manifolds are also possible, i.e. it is enough to assume a non-compact, smooth 4-manifold endowed with a codimension-1 foliation.

All these restrictions on the representation of spacetime by the manifold concept are clearly motivated by physical questions. Among the properties there is one distinguished element: the smoothness. Usually one assumes a smooth, unique atlas of charts (i.e. a smooth or differential structure) covering the manifold where the smoothness is induced by the unique smooth structure on \mathbb{R} . But that is not the full story. Even in dimension 4, there are an infinity of possible other smoothness structures (i.e. a smooth atlas) non-diffeomorphic to each other. For a deeper insight we refer to the book [19].

2.1. Exotic 4-manifolds and exotic \mathbb{R}^4

If two manifolds are homeomorphic but non-diffeomorphic, they are **exotic** to each other. The smoothness structure is called an **exotic smoothness structure**.

The implications for physics are tremendous because we rely on the smooth calculus to formulate field theories. Thus different smoothness structures have to represent different physical situations leading to different measurable results. But it should be stressed that *exotic smoothness is not exotic physics*. Exotic smoothness is part of topology (differential topology), i.e. a finer determination of the topology (the smooth atlas of the manifold) fulfilling the condition of smoothness. Therefore we obtain another parameter to vary a manifold with fixed topology. Usually one starts with a topological manifold M and introduces structures on them. Then one has the following ladder of possible structures:

$$\begin{aligned} \text{Topology} &\rightarrow \text{piecewise-linear(PL)} \rightarrow \text{Smoothness} \\ &\rightarrow \text{bundles, Lorentz, Spin etc.} \rightarrow \text{metric, ...} \end{aligned}$$

We do not want to discuss the first transition, i.e. the existence of a triangulation on a topological manifold. But we remark that the existence of a PL structure implies uniquely a smoothness structure in all dimensions smaller than 7 [20].

Given two homeomorphic manifolds M, M' , how can we compare both manifolds to decide whether both are diffeomorphic? A mapping between two manifolds M, M' can be described by a $(n + 1)$ -dimensional manifold W with $\partial W = M \sqcup M'$, called a cobordism W . In the following, the two homeomorphic manifolds M, M' are simple-connected. The celebrated h-cobordism theorem of Smale [21, 22] gives a simple criteria when a diffeomorphism between M and M' for dimension greater than 5 exists: there is a cobordism W between M and M' where the inclusions $M, M' \hookrightarrow W$ induce homotopy-equivalences between M, M' and W . We call W a h-cobordism. Therefore the classification problem of smoothness structures in higher dimensions (> 5) is a homotopy-theoretic problem. The set of smooth structures (up to h-cobordisms) of the n -sphere S^n has the structure of an abelian group Θ_n (under connected sum $\#$). Via the h-cobordism theorem this group is identical to the group of homotopy spheres which was analyzed by Kervaire and Milnor [23] to be a finite group. Later on the result about the exotic spheres can be extended to any smoothable manifold [20], i.e. there is only a finite number of non-diffeomorphic smoothness structures. In all dimensions smaller than four, there is an unique smoothness structure, the standard structure. But all methods failed for the special case of dimension four.

Now we consider two homeomorphic, smooth, but non-diffeomorphic 4-manifolds M_0 and M . As expressed above, a comparison of both smoothness structures is given by a h-cobordism W between M_0 and M (M, M_0 are homeomorphic). Let the 4-manifolds additionally be compact, closed and simple-connected, then we have the structure theorem¹ of h-cobordisms [24]:

Theorem 1 *Let W be a h-cobordism between M_0, M , then there are contractable submanifolds $A_0 \subset M_0, A \subset M$ and a h-subcobordism $X \subset W$ with $\partial X = A_0 \sqcup A$, so that the remaining h-cobordism $W \setminus X$ trivializes $W \setminus X = (M_0 \setminus A_0) \times [0, 1]$ inducing a diffeomorphism between $M_0 \setminus A_0$ and $M \setminus A$.*

In short that means that the smoothness structure of M is determined by the contractable manifold A – its Akbulut cork – and by the embedding of A into M . As shown by Freedman[25], the Akbulut cork has a homology 3-sphere² as boundary. The embedding of the cork can be derived now from the structure of the h-subcobordism X between A_0 and A . For that purpose we cut A_0 out from M_0 and A out from M . Then we glue in both submanifolds A_0, A via the maps

¹ A diffeomorphism will be described by the symbol $=$ in the following.

² A homology 3-sphere is a 3-manifold with the same homology as the 3-sphere S^3 .

$\tau_0 : \partial A_0 \rightarrow \partial(M_0 \setminus A_0) = \partial A_0$ and $\tau : \partial A \rightarrow \partial(M \setminus A) = \partial A$. Both maps τ_0, τ are involutions, i.e. $\tau \circ \tau = id$. One of these maps (say τ_0) can be chosen to be trivial (say $\tau_0 = id$). Thus the involution τ determines the smoothness structure. Especially the topology of the Akbulut cork A and its boundary ∂A is given by the topology of M . For instance, the Akbulut cork of the blow-uped 4-dimensional $K3$ surface $K3 \# \overline{CP}^2$ is the so-called *Mazur manifold* [26, 27] with the *Brieskorn-Sphere* $B(2, 5, 7)$ as boundary. Akbulut and its coworkers [28, 29] discuss many examples of Akbulut corks and the dependence of the smoothness structure on the cork.

Then as shown by Bizaca and Gompf [30] the neighborhood N of the handle pair as well the neighborhood $N(A)$ of the embedded Akbulut cork consists of the cork A and the Casson handle CH . Especially the open neighborhood $N(A)$ of the Akbulut cork is an exotic \mathbb{R}^4 . The situation was analyzed in [31]:

Theorem 2 *Let W^5 be a non-trivial (smooth) h-cobordism between M_0^4 and M^4 (i.e. W is not diffeomorphic to $M \times [0, 1]$). Then there is an open sub-h-cobordism U^5 that is homeomorphic to $\mathbb{R}^4 \times [0, 1]$ and contains a compact contractable sub-h-cobordism X (the cobordism between the Akbulut corks, see above), such that both W and U are trivial cobordisms outside of X , i.e. one has the diffeomorphisms*

$$\begin{aligned} W \setminus X &= ((W \cap M) \setminus X) \times [0, 1] & \text{and} \\ U \setminus X &= ((U \cap M) \setminus X) \times [0, 1] \end{aligned}$$

(the latter can be chosen to be the restriction of the former). Furthermore the open sets $U \cap M$ and $U \cap M_0$ are homeomorphic to \mathbb{R}^4 which are exotic \mathbb{R}^4 if W is non-trivial.

Then one gets an exotic \mathbb{R}^4 which smoothly embeds automatically in the 4-sphere, called a small exotic \mathbb{R}^4 . Furthermore we remark that the exoticness of the \mathbb{R}^4 is connected with the non-trivial smooth h-cobordism W^5 , i.e. the failure of the smooth h-cobordism theorem implies the existence of small exotic \mathbb{R}^4 's.

Let R be a small exotic \mathbb{R}^4 induced from the non-product h-cobordism W between M and M_0 with Akbulut corks $A \subset M$ and $A_0 \subset M_0$, respectively. Let $K \subset \mathbb{R}^4$ be a compact subset. Bizaca and Gompf [30] constructed the small exotic \mathbb{R}_1^4 by using the simplest tree $Tree_+$. Bizaca [12, 32] showed that the Casson handle generated by $Tree_+$ is an exotic Casson handle. Using Theorem 3.2 of [33], there is a topological radius function $\rho : \mathbb{R}_1^4 \rightarrow [0, +\infty)$ (polar coordinates) so that $\mathbb{R}_t^4 = \rho^{-1}([0, r))$ with $t = 1 - \frac{1}{r}$. Then $K \subset \mathbb{R}_0^4$ and \mathbb{R}_t^4 is also a small exotic \mathbb{R}^4 for t belonging to a Cantor set $CS \subset [0, 1]$. Especially two exotic \mathbb{R}_s^4 and \mathbb{R}_t^4 are non-diffeomorphic for $s < t$ except for countable many pairs. In [33] it was claimed that there is a smoothly embedded homology 4-disk A . The boundary ∂A is a homology 3-sphere with a non-trivial representation of its fundamental group into $SO(3)$ (so ∂A cannot be diffeomorphic to a 3-sphere). According to Theorem 2 this homology 4-disk must be identified with the Akbulut cork of the non-trivial h-cobordism. The cork A is contractable and can be (at least) build by one 1-handle and one 2-handle (case of a Mazur manifold). Given a radial family \mathbb{R}_t^4 with radius $r = \frac{1}{1-t}$ so that $t = 1 - \frac{1}{r} \in CS \subset [0, 1]$. Suppose there is a diffeomorphism

$$(d, id_K) : (\mathbb{R}_s^4, K) \rightarrow (\mathbb{R}_t^4, K) \quad s \neq t \in CS$$

fixing the compact subset K . Then this map d induces end-periodic manifolds³ $M \setminus (\bigcap_{i=0}^{\infty} d^i(\mathbb{R}_s^4))$ and $M_0 \setminus (\bigcap_{i=0}^{\infty} d^i(\mathbb{R}_s^4))$ which must be smoothable contradicting a theorem of Taubes [34]. Therefore \mathbb{R}_s^4 and \mathbb{R}_t^4 are non-diffeomorphic for $t \neq s$ (except for countable many possibilities).

³ We ignore the inclusion for simplicity.

2.2. Exotic \mathbb{R}^4 and codimension-1 foliations

In this subsection we will construct a codimension-one foliation on the boundary ∂A of the cork with non-trivial Godbillon-Vey invariant.

A codimension k foliation⁴ of an n -manifold M^n (see the nice overview article [35]) is a geometric structure which is formally defined by an atlas $\{\phi_i : U_i \rightarrow M^n\}$, with $U_i \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$, such that the transition functions have the form

$$\phi_{ij}(x, y) = (f(x, y), g(y)), \quad \left[x \in \mathbb{R}^{n-k}, y \in \mathbb{R}^k \right] \quad .$$

Intuitively, a foliation is a pattern of $(n - k)$ -dimensional stripes - i.e., submanifolds - on M^n , called the leaves of the foliation, which are locally well-behaved. The tangent space to the leaves of a foliation \mathcal{F} forms a vector bundle over M^n , denoted $T\mathcal{F}$. The complementary bundle $\nu\mathcal{F} = TM^n/T\mathcal{F}$ is the normal bundle of \mathcal{F} . Such foliations are called regular in contrast to singular foliations or Haefliger structures. For the important case of a codimension-1 foliation we need an overall non-vanishing vector field or its dual, an one-form ω . This one-form defines a foliation iff it is integrable, i.e.

$$d\omega \wedge \omega = 0 \quad (1)$$

The cross-product $M \times N$ defines for example a trivial foliation. Now we will discuss an important equivalence relation between foliations, cobordant foliations.

Definition 1 *Let M_0 and M_1 be two closed, oriented m -manifolds with codimension- q foliations. Then these foliated manifolds are said to be foliated cobordant if there is a compact, oriented $(m+1)$ -manifold with boundary $\partial W = M_0 \sqcup \overline{M}_1$ and with a codimension- q foliation \mathcal{F} transverse to the boundary. Every leaf L_α of the foliation \mathcal{F} induces leafs $L_\alpha \cap \partial W$ of foliations $\mathcal{F}_{M_0}, \mathcal{F}_{M_1}$ on the two components of the boundary ∂W .*

The resulted foliated cobordism classes $[\mathcal{F}_M]$ of the manifold M form an abelian group $\mathcal{CF}_{m,q}(M)$ under disjoint union \sqcup (inverse \overline{M} , unit $S^q \times S^{m-q}$, see [36] §29).

In [37], Thurston constructed a foliation of the 3-sphere S^3 depending on a polygon P in the hyperbolic plane \mathbb{H}^2 so that two foliations are non-cobordant if the corresponding polygons have different areas. Now we consider two codimension-1 foliations $\mathcal{F}_1, \mathcal{F}_2$ depending on the convex polygons P_1 and P_2 in \mathbb{H}^2 . As mentioned above, these foliations $\mathcal{F}_1, \mathcal{F}_2$ are defined by two one-forms ω_1 and ω_2 with $d\omega_a \wedge \omega_a = 0$ and $a = 0, 1$. Now we define the one-forms θ_a as the solution of the equation

$$d\omega_a = -\theta_a \wedge \omega_a \quad (2)$$

and consider the closed 3-form

$$\Gamma_{\mathcal{F}_a} = \theta_a \wedge d\theta_a \quad (3)$$

associated to the foliation \mathcal{F}_a . As discovered by Godbillon and Vey [38], $\Gamma_{\mathcal{F}}$ depends only on the foliation \mathcal{F} and not on the realization via ω, θ . Thus $\Gamma_{\mathcal{F}}$, the *Godbillon-Vey class*, is an invariant of the foliation. Let \mathcal{F}_1 and \mathcal{F}_2 be two cobordant foliations then $\Gamma_{\mathcal{F}_1} = \Gamma_{\mathcal{F}_2}$. Thurston [37] obtains for the Godbillon-Vey number

$$GV(S^3, \mathcal{F}_a) = \langle \Gamma_{\mathcal{F}_a}, [S^3] \rangle = \int_{S^3} \Gamma_{\mathcal{F}_a} = \text{vol}(\pi^{-1}(Q)) = 4\pi \cdot \text{Area}(P_a) \quad (4)$$

where only the foliation on $M = (S^2 \setminus \{k \text{ punctures}\}) \times S^1$ contributes to the Godbillon-Vey class because the Reeb foliations have vanishing Godbillon-Vey class. Let $[1] \in H^3(S^3, \mathbb{R})$ be

⁴ In general, the differentiability of a foliation is very important. Here we consider the smooth case only.

Table 1. relation between foliation and operator algebra

Foliation	Operator algebra
leaf	operator
closed curve transversal to foliation	projector (idempotent operator)
holonomy	linear functional (state)
local chart	center of algebra

the dual of the fundamental class $[S^3]$ defined by the volume form, then the Godbillon-Vey class can be represented by

$$\Gamma_{\mathcal{F}_a} = 4\pi \cdot \text{Area}(P_a)[1] \quad (5)$$

Furthermore the Godbillon-Vey number GV seen as linear functional defines a surjective homomorphism

$$GV : \mathcal{CF}_{3,1}(S^3) \rightarrow \mathbb{R} \quad (6)$$

from the group of foliated cobordisms $\mathcal{CF}_{3,1}(S^3)$ of the 3-sphere to the real numbers of possible areas $\text{Area}(P)$, i.e.

- \mathcal{F}_1 is cobordant to $\mathcal{F}_2 \implies \text{Area}(P_1) = \text{Area}(P_2)$ (the reverse direction (injectivity) is open)
- \mathcal{F}_1 and \mathcal{F}_2 are non-cobordant $\iff \text{Area}(P_1) \neq \text{Area}(P_2)$

We note that $\text{Area}(P) = (k-2)\pi - \sum_k \alpha_k$. The Godbillon-Vey class is an element of the deRham cohomology $H^3(S^3, \mathbb{R})$ which will be used later to construct a relation to gerbes. Furthermore we remark that the classification is not complete. Thurston constructed only a surjective homomorphism from the group of cobordism classes of foliation of S^3 into the real numbers \mathbb{R} . We remark the close connection between the Godbillon-Vey class (3) and the Chern-Simons form if θ can be interpreted as connection of a suitable line bundle.

In [13] we presented the relation between foliations on the boundary ∂A of the cork A and the radial family of small exotic \mathbb{R}^4 . Especially we have:

Theorem 3 *Given a radial family R_t of small exotic \mathbb{R}_t^4 with radius r and $t = 1 - \frac{1}{r} \in CS \subset [0, 1]$ induced from the non-product h -cobordism W between M and M_0 with Akbulut cork $A \subset M$ and $A \subset M_0$, respectively. The radial family R_t determines a family of codimension-one foliations of ∂A with Godbillon-Vey number r^2 . Furthermore given two exotic spaces R_t and R_s , homeomorphic but non-diffeomorphic to each other (and so $t \neq s$) then the two corresponding codimension-one foliation of ∂A are non-cobordant to each other.*

2.3. From exotic smoothness to operator algebras

Connes [39] constructed the operator algebra $C_r^*(M, F)$ associated to a foliation F . The correspondence between a foliation and the operator algebra (as well as the von Neumann algebra) is visualized by table 1. As extract of our previous paper [13], we obtained a relation between exotic \mathbb{R}^4 's and codimension-1 foliations of the 3-sphere S^3 . For a codimension-1 foliation there is the Godbillon-Vey invariant [38] as element of $H^3(M, \mathbb{R})$. Hurder and Katok [40] showed that the C^* -algebra of a foliation with non-trivial Godbillon-Vey invariant contains a factor III subalgebra. Using Tomita-Takesaki-theory, one has a continuous decomposition (as crossed product) of any factor III algebra M into a factor II_∞ algebra N together with a one-parameter group⁵ $(\theta_\lambda)_{\lambda \in \mathbb{R}_+^*}$ of automorphisms $\theta_\lambda \in \text{Aut}(N)$ of N , i.e. one obtains

⁵ The group \mathbb{R}_+^* is the group of positive real numbers with multiplication as group operation also known as Pontrjagin dual.

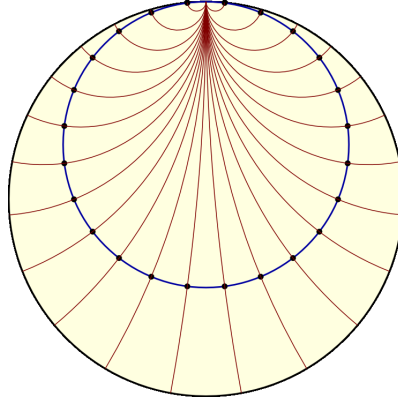


Figure 1. horocycle, a curve whose normals all converge asymptotically

$$M = N \rtimes_{\theta} \mathbb{R}_+^* .$$

But that means, there is a foliation induced from the foliation of the S^3 producing this II_{∞} factor. Connes [39] (in section I.4 page 57ff) constructed the foliation F' canonically associated to F having the factor II_{∞} as von Neumann algebra. In our case it is the horocycle flow: Let P the polygon on the hyperbolic space \mathbb{H}^2 determining the foliation of the S^3 (see above). P is equipped with the hyperbolic metric $2|dz|/(1 - |z|^2)$ together with the collection T_1P of unit tangent vectors to P . A horocycle in P is a circle contained in P which touches ∂P at one point (see Fig. 1). Then the horocycle flow $T_1P \rightarrow T_1P$ is the flow moving an unit tangent vector along a horocycle (in positive direction at unit speed). Then the polygon P determines a surface S of genus $g > 1$ with abelian torsion-less fundamental group $\pi_1(S)$ so that the homomorphism $\pi_1(S) \rightarrow \mathbb{R}$ determines an unique (ergodic invariant) Radon measure. Finally the horocycle flow determines a factor II_{∞} foliation associated to the factor III_1 foliation. We remark for later usage that this foliation is determined by a set of closed curves (the horocycles). Furthermore the idempotent operators in the operator algebra $C_r^*(M, F)$ of the foliation F are represented by closed curves transversal to the foliation (see [39]).

3. Quantization

In this section we describe a deep relation between quantization and the codimension-1 foliation of the S^3 determining the smoothness structure on a small exotic \mathbb{R}^4 . Here and in the following *we will identify the leaf space with its operator algebra*.

3.1. The observable algebra and Poisson structure

In this section we will describe the formal structure of a classical theory coming from the algebra of observables using the concept of a Poisson algebra. In quantum theory, an observable is represented by a hermitean operator having the spectral decomposition via projectors or idempotent operators. The coefficient of the projector is the eigenvalue of the observable or one possible result of a measurement. At least one of these projectors represent (via the GNS representation) a quasi-classical state. Thus to construct the substitute of a classical observable algebra with Poisson algebra structure we have to concentrate on the idempotents in the C^* algebra. Now we will see that the set of closed curves on a surface has the structure of a Poisson algebra. Let us start with the definition of a Poisson algebra.

Definition 2 Let P be a commutative algebra with unit over \mathbb{R} or \mathbb{C} . A Poisson bracket on P is a bilinearform $\{, \} : P \otimes P \rightarrow P$ fulfilling the following 3 conditions:

anti-symmetry $\{a, b\} = -\{b, a\}$

Jacobi identity $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$

derivation $\{ab, c\} = a\{b, c\} + b\{a, c\}$.

Then a Poisson algebra is the algebra $(P, \{, \})$.

Now we consider a surface S together with a closed curve γ . Additionally we have a Lie group G given by the isometry group. The closed curve is one element of the fundamental group $\pi_1(S)$. From the theory of surfaces we know that $\pi_1(S)$ is a free abelian group. Denote by Z the free \mathbb{K} -module (\mathbb{K} a ring with unit) with the basis $\pi_1(S)$, i.e. Z is a freely generated \mathbb{K} -modul. Recall Goldman's definition of the Lie bracket in Z (see [41]). For a loop $\gamma : S^1 \rightarrow S$ we denote its class in $\pi_1(S)$ by $\langle \gamma \rangle$. Let α, β be two loops on S lying in general position. Denote the (finite) set $\alpha(S^1) \cap \beta(S^1)$ by $\alpha \# \beta$. For $q \in \alpha \# \beta$ denote by $\epsilon(q; \alpha, \beta) = \pm 1$ the intersection index of α and β in q . Denote by $\alpha_q \beta_q$ the product of the loops α, β based in q . Up to homotopy the loop $(\alpha_q \beta_q)(S^1)$ is obtained from $\alpha(S^1) \cup \beta(S^1)$ by the orientation preserving smoothing of the crossing in the point q . Set

$$[\langle \alpha \rangle, \langle \beta \rangle] = \sum_{q \in \alpha \# \beta} \epsilon(q; \alpha, \beta) (\alpha_q \beta_q) \quad . \quad (7)$$

According to Goldman [41], Theorem 5.2, the bilinear pairing $[\cdot, \cdot] : Z \times Z \rightarrow Z$ given by (7) on the generators is well defined and makes Z to a Lie algebra. The algebra $Sym(Z)$ of symmetric tensors is then a Poisson algebra (see Turaev [42]).

Now we introduce a principal G bundle on S , representing a geometry on the surface. This bundle is induced from a G bundle over $S \times [0, 1]$ having always a flat connection. Alternatively one can consider a homomorphism $\pi_1(S) \rightarrow G$ represented as holonomy functional

$$hol(\omega, \gamma) = \mathcal{P} \exp \left(\int_{\gamma} \omega \right) \in G \quad (8)$$

with the path ordering operator \mathcal{P} and ω as flat connection (i.e. inducing a flat curvature $\Omega = d\omega + \omega \wedge \omega = 0$). This functional is unique up to conjugation induced by a gauge transformation of the connection. Thus we have to consider the conjugation classes of maps

$$hol : \pi_1(S) \rightarrow G$$

forming the space $X(S, G)$ of gauge-invariant flat connections of principal G bundles over S . Then [43] constructed of the Poisson structure on $X(S, G)$, i.e. the space $X(S, G)$ has a natural Poisson structure (induced by the bilinear form (7) on the group) and the Poisson algebra $(X(S, G), \{, \})$ of complex functions over them is the algebra of observables. For the following we will fix the group to be $G = SL(2, \mathbb{C})$ as the largest isometry group of a homogeneous 3-manifold (or space of constant curvature). The space $X(S, SL(2, \mathbb{C}))$ has a natural Poisson structure (induced by the bilinear form of Goldman [41] on the group) and the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$ of complex functions over them is the algebra of observables.

3.2. Drinfeld-Turaev Quantization

Now we introduce the ring $\mathbb{C}[[h]]$ of formal polynomials in h with values in \mathbb{C} . This ring has a topological structure, i.e. for a given power series $a \in \mathbb{C}[[h]]$ the set $a + h^n \mathbb{C}[[h]]$ forms a neighborhood. Now we define

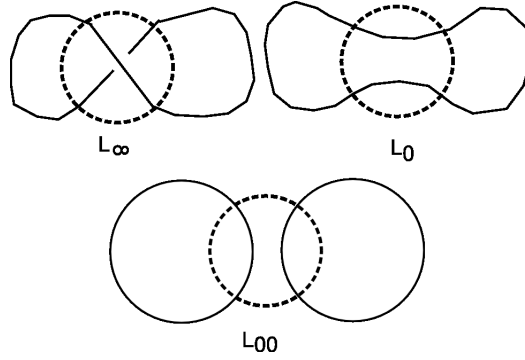


Figure 2. crossings L_∞, L_o, L_{oo}

Definition 3 A Quantization of a Poisson algebra P is a $\mathbb{C}[[h]]$ algebra P_h together with the \mathbb{C} -algebra isomorphism $\Theta : P_h/hP \rightarrow P$ so that

1. the modul P_h is isomorphic to $V[[h]]$ for a \mathbb{C} vector space V
2. let $a, b \in P$ and $a', b' \in P_h$ be $\Theta(a) = a', \Theta(b) = b'$ then

$$\Theta \left(\frac{a'b' - b'a'}{h} \right) = \{a, b\}$$

One speaks of a deformation of the Poisson algebra by using a deformation parameter h to get a relation between the Poisson bracket and the commutator. Therefore we have the problem to find the deformation of the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$. The solution to this problem can be found via two steps:

- (i) at first find another description of the Poisson algebra by a structure with one parameter at a special value and
- (ii) secondly vary this parameter to get the deformation.

Fortunately both problems were already solved (see [44, 42]). The solution of the first problem is :

The Skein modul $K_{-1}(S \times [0, 1])$ (i.e. $t = -1$) has the structure of an algebra isomorphic to the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$. (see [45, 46])

Then we have also the solution of the second problem:

The skein algebra $K_t(S \times [0, 1])$ is the quantization of the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{, \})$ with the deformation parameter $t = \exp(h/4)$. (see [45])

To understand these solutions we have to introduce the skein module $K_t(M)$ of a 3-manifold M (see [47]). For that purpose we consider the set of links $\mathcal{L}(M)$ in M up to isotopy and construct the vector space $\mathbb{C}\mathcal{L}(M)$ with basis $\mathcal{L}(M)$. Then one can define $\mathbb{C}\mathcal{L}[[t]]$ as ring of formal polynomials having coefficients in $\mathbb{C}\mathcal{L}(M)$. Now we consider the link diagram of a link, i.e. the projection of the link to the \mathbb{R}^2 having the crossings in mind. Choosing a disk in \mathbb{R}^2 so that one crossing is inside this disk. If the three links differ by the three crossings L_{oo}, L_o, L_∞ (see figure 2) inside of the disk then these links are skein related. Then in $\mathbb{C}\mathcal{L}[[t]]$ one writes the skein relation⁶ $L_\infty - tL_o - t^{-1}L_{oo}$. Furthermore let $L \sqcup O$ be the disjoint union of the link with a circle then one writes the framing relation $L \sqcup O + (t^2 + t^{-2})L$. Let $S(M)$ be the smallest submodule of $\mathbb{C}\mathcal{L}[[t]]$ containing both relations, then we define the Kauffman bracket skein modul by $K_t(M) = \mathbb{C}\mathcal{L}[[t]]/S(M)$. We list the following general results about this modul:

⁶ The relation depends on the group $SL(2, \mathbb{C})$.

- The modul $K_{-1}(M)$ for $t = -1$ is a commutative algebra.
- Let S be a surface then $K_t(S \times [0, 1])$ carries the structure of an algebra.

The algebra structure of $K_t(S \times [0, 1])$ can be simply seen by using the diffeomorphism between the sum $S \times [0, 1] \cup_S S \times [0, 1]$ along S and $S \times [0, 1]$. Then the product ab of two elements $a, b \in K_t(S \times [0, 1])$ is a link in $S \times [0, 1] \cup_S S \times [0, 1]$ corresponding to a link in $S \times [0, 1]$ via the diffeomorphism. The algebra $K_t(S \times [0, 1])$ is in general non-commutative for $t \neq -1$. For the following we will omit the interval $[0, 1]$ and denote the skein algebra by $K_t(S)$. Furthermore we remark, that *all results remain true if we use an intersection in L_∞ instead of an over- or undercrossing.*

Ad hoc the skein algebra is not directly related to the foliation. We used only the fact that there is an idempotent in the C^* algebra represented by a closed curve. It is more satisfying to obtain a direct relation between both constructions. Then the von Neumann algebra of the foliation is the result of a quantization in the physical sense.

With more care (see [14]) we can construct a relation of the skein algebra to the Temperley-Lieb algebra seen as a subalgebra of the hyperfinite factor III_1 algebra. Then we have a closed circle:

- (i) The Thurston foliation F of the S^3 is associated to the hyperfinite factor III_1 algebra.
- (ii) Using Tomita-Takesaki-theory, one has a continuous decomposition (as crossed product) of any factor III algebra M into a factor II_∞ algebra N together with a one-parameter group $(\theta_\lambda)_{\lambda \in \mathbb{R}_+^*}$ of automorphisms $\theta_\lambda \in \text{Aut}(N)$ of N . The factor II_∞ algebra is isomorphic to $II_1 \otimes I_\infty$.
- (iii) As Jones [48] showed, the factor II_1 is given by an infinite sequence of Temperley-Lieb algebras.
- (iv) One can construct the foliation F' , the horocycle flow of $T_1 S$ of the surface S , with factor II_∞ algebra related to F .
- (v) The skein algebra $K_t(S)$ represents the horocycle flow foliation isomorphic to the Temperley-Lieb algebra or the factor II_1 .
- (vi) But the skein algebra is the quantization of a Poisson algebra given by complex functions over $X(S, SL(2, \mathbb{C}))$. Therefore the operator algebra of the foliation F' (and also of F) comes from the quantization of a classical Poisson algebra (via deformation quantization).
- (vii) One of the central elements in the algebraic quantum field theory is the Tomita-Takesaki theory leading to the III_1 factor as vacuum sector [16]. This factor can be now constructed by using the foliation induced by an exotic small \mathbb{R}^4 .

4. Quantum field theory (QFT)

In this paper we will understand a relativistic QFT as an algebraic QFT (AQFT) in the sense of Haag-Kastler (Local quantum field theory) [15]. A series of results, accumulated over a period of more than thirty years, indicates that the local algebras of relativistic QFT are type III von Neumann algebras, and more specifically, hyperfinite type III_1 factors. Therefore our relation between exotic \mathbb{R}^4 and this factor is a crucial step in the understanding of QFT from the geometrical point of view. Any realistic QFT starts with an action to fix the fields. Here we will discuss a method to derive this action using the structure of the exotic \mathbb{R}^4 (see [49]).

We will start with some remarks. After the work of Witten [50] on topological QFTs, there was a growing interest in the relation between QFT and topology. Witten proposed a supersymmetric QFT to obtain the Donaldson polynomials as expectation values. Then the further development of this approach led directly to the Seiberg-Witten invariants [51, 52, 53]. Therefore one would expect a strong relation between QFT and 4-dimensional differential topology. Our approach above gives some hint in this direction.

The interpretation of this relation is given by the infinite constructions in 4-manifold theory like Casson handle [54, 55, 25] or capped gropes [56, 57] used in the construction of the relation between codimension-1 foliations and exotic \mathbb{R}^4 . All these structures are described by trees. For example a Casson handle CH is specified up to (orientation preserving) diffeomorphism (of pairs) by a labeled finitely-branching tree with base-point $*$, having all edge paths infinitely extendable away from $*$. Each edge should be given a label $+$ or $-$. Here is the construction: $\text{tree} \rightarrow CH$. Each vertex corresponds to a kinky handle; the self-plumbing number of that kinky handle equals the number of branches leaving the vertex. The sign on each branch corresponds to the sign of the associated self plumbing. The whole process generates a tree with infinite many levels. In principle, every tree with a finite number of branches per level realizes a corresponding Casson handle. The Casson handle is at the heart of small exotic \mathbb{R}^4 (see [30]).

Therefore we have to understand one level of the tree. Then the infinite repetition of the levels forms the tree. But the basic structure in QFT is also given by a tree expressing the renormalized fields by the forest formula [58]. So we start with the basic structure of the Casson handle forming the levels of the tree in 4-manifold theory, the immersed disk. Now we will follow the paper [59] very closely. Let D^2 be a 2-disk and M a 4-manifold as spacetime. The map $i : D^2 \rightarrow M$ is called an immersion if the differential $di : TD^2 \rightarrow TM$ is injective. It is known from singularity theory [60] that every map of a 2-manifold into a 4-manifold can be deformed to an immersion, the immersion may not be an embedding i.e. the immersed disk may have self-intersections. For the following discussion we consider the immersion $D^2 \rightarrow U \subset \mathbb{R}^4$ of the disk into one chart U of M .

For simplicity, start with a toy model of an immersion of a surface into the 3-dimensional Euclidean space. Let $f : M^2 \rightarrow \mathbb{R}^3$ be a smooth map of a Riemannian surface with injective differential $df : TM^2 \rightarrow T\mathbb{R}^3$, i.e. an immersion. In the *Weierstrass representation* one expresses a *conformal minimal* immersion f in terms of a holomorphic function $g \in \Lambda^0$ and a holomorphic 1-form $\mu \in \Lambda^{1,0}$ as the integral

$$f = \text{Re} \left(\int (1 - g^2, i(1 + g^2), 2g)\mu \right) .$$

An immersion of M^2 is conformal if the induced metric g on M^2 has components

$$g_{zz} = 0 = g_{\bar{z}\bar{z}}, \quad g_{z\bar{z}} \neq 0$$

and it is minimal if the surface has minimal volume. Now we consider a spinor bundle S on M^2 (i.e. $TM^2 = S \otimes S$ as complex line bundles) and with the splitting

$$S = S^+ \oplus S^- = \Lambda^0 \oplus \Lambda^{1,0}$$

Therefore the pair (g, μ) can be considered as spinor field φ on M^2 . Then the Cauchy-Riemann equation for g and μ is equivalent to the Dirac equation $D\varphi = 0$. The generalization from a conformal minimal immersion to a conformal immersion was done by many authors (see the references in [59]) to show that the spinor φ now fulfills the Dirac equation

$$D\varphi = K\varphi \tag{9}$$

where K is the mean curvature (i.e. the trace of the second fundamental form). The minimal case is equivalent to the vanishing mean curvature $H = 0$ recovering the equation above. Friedrich [59] uncovered the relation between a spinor Φ on \mathbb{R}^3 and the spinor $\varphi = \Phi|_{M^2}$: if the spinor Φ fulfills the Dirac equation $D\Phi = 0$ then the restriction $\varphi = \Phi|_{M^2}$ fulfills equation (9) and $|\varphi|^2 = \text{const.}$ Therefore we obtain

$$H = \bar{\varphi}D\varphi \tag{10}$$

with $|\varphi|^2 = 1$.

Now we will discuss the more complicated case. For that purpose we consider the kinky handle which can be seen as the image of an immersion $I : D^2 \times D^2 \rightarrow \mathbb{R}^4$. This map determines a restriction of the immersion $I|_{\partial} : \partial D^2 \times D^2 \rightarrow \mathbb{R}^4$ with image a knotted solid torus $T(K) = I|_{\partial}(\partial D^2 \times D^2)$. But a knotted solid torus $T(K) = K \times D^2$ is uniquely determined by its boundary $\partial T(K) = K \times \partial D^2 = K \times S^1$, a knotted torus given as image $\partial T(K) = I|_{\partial \times \partial}(T^2)$ of the immersion $I|_{\partial \times \partial} : T^2 = S^1 \times S^1 \rightarrow \mathbb{R}^3$. But as discussed above, this immersion $I|_{\partial \times \partial}$ can be defined by a spinor φ on T^2 fulfilling the Dirac equation

$$D\varphi = H\varphi \quad (11)$$

with $|\varphi|^2 = 1$ (or an arbitrary constant) (see Theorem 1 of [59]). The transition to the case of the immersion $I|_{\partial}$ can be done by constructing a spinor ϕ out of φ which is constant along the normal of the immersed torus T^2 . As discussed above a spinor bundle over a surface splits into two sub-bundles $S = S^+ \oplus S^-$ with the corresponding splitting of the spinor φ in components

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$$

and we have the Dirac equation

$$D\varphi = \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix} = H \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}$$

with respect to the coordinates (z, \bar{z}) on T^2 . In dimension 3 we have a spinor bundle of same fiber dimension then the spin bundle S but without a splitting into two sub-bundles. Now we define the extended spinor ϕ over the solid torus $\partial D^2 \times D^2$ via the restriction $\phi|_{T^2} = \varphi$. Then ϕ is constant along the normal vector $\partial_N \phi = 0$ fulfilling the 3-dimensional Dirac equation

$$D^{3D}\phi = \begin{pmatrix} \partial_N & \partial_z \\ \partial_{\bar{z}} & -\partial_N \end{pmatrix} \phi = H\phi \quad (12)$$

induced from the Dirac equation (11) via restriction and where $|\phi|^2 = \text{const.}$ Especially we obtain for the mean curvature

$$H = \bar{\phi} D^{3D} \phi \quad (13)$$

of the knotted solid torus $T(K)$ (up to a constant from $|\phi|^2$). Or in local coordinates

$$H = \bar{\phi} \sigma^\mu D_\mu^{3D} \phi \quad (14)$$

with the Pauli matrices σ^μ . Thus the level of the tree is described by the mean curvature H of the knotted solid torus $T(K)$ or invariantly by the integral

$$\int_{T(K)} H_{T(K)} \sqrt{g_\partial} d^3x = \int_{T(K)} \psi \gamma^\mu D_\mu \bar{\psi} \sqrt{g_\partial} d^3x \quad (15)$$

with the metric g_∂ at $T(K)$, i.e. by the Dirac action. Now we will discuss the extension from the 3D to the 4D case. Let $\iota : D^2 \times S^1 \hookrightarrow M$ be an immersion of the solid torus $\Sigma = D^2 \times S^1$ into the 4-manifold M with the normal vector \vec{N} . The spin bundle S_M of the 4-manifold splits into two sub-bundles S_M^\pm where one subbundle, say S_M^+ , can be related to the spin bundle S_Σ . Then the spin bundles are related by $S_\Sigma = \iota^* S_M^+$ with the same relation $\phi = \iota_* \Phi$ for the spinors

($\phi \in \Gamma(S_\Sigma)$ and $\Phi \in \Gamma(S_M^+)$). Let $\nabla_X^M, \nabla_X^\Sigma$ be the covariant derivatives in the spin bundles along a vector field X as section of the bundle $T\Sigma$. Then we have the formula

$$\nabla_X^M(\Phi) = \nabla_X^\Sigma \phi - \frac{1}{2}(\nabla_X \vec{N}) \cdot \vec{N} \cdot \phi \quad (16)$$

with the obvious embedding $\phi \mapsto \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \Phi$ of the spinor spaces. The expression $\nabla_X \vec{N}$ is the second fundamental form of the immersion with trace the mean curvature $2H$. Then from (16) one obtains a similar relation between the corresponding Dirac operators

$$D^M \Phi = D^{3D} \phi - H \phi \quad (17)$$

with the Dirac operator D^{3D} defined via (12). Together with equation (12) we obtain

$$D^M \Phi = 0 \quad (18)$$

i.e. Φ is a parallel spinor.

Conclusion: There is a relation between a 3-dimensional spinor ϕ on a 3-manifold Σ fulfilling a Dirac equation $D^\Sigma \phi = H \phi$ (determined by the immersion $\Sigma \rightarrow M$ into a 4-manifold M) and a 4-dimensional spinor Φ on a 4-manifold M with fixed chirality ($\in \Gamma(S_M^+)$ or $\in \Gamma(S_M^-)$) fulfilling the Dirac equation $D^M \Phi = 0$.

From the Dirac equation (18) we obtain the the action

$$\int_M \bar{\Phi} D^M \Phi \sqrt{g} d^4 x$$

as an extension of (15) to the whole spacetime M . By variation of the action (15) we obtain an immersion of minimal mean curvature, i.e. $H = 0$. Then we can identify via relation (17) the 4-dimensional and the 3-dimensional action via

$$S_F(M) = \int_M \bar{\Phi} D^M \Phi \sqrt{g_M} d^4 x = \int_{T(K)} \bar{\phi} D^{3D} \phi \sqrt{g_\partial} d^3 x = \int_{T(K)} H_{T(K)} \sqrt{g_\partial} d^3 x$$

Therefore the 3-dimensional action (15) can be extended to the whole 4-manifold (but for a spinor Φ of fixed chirality). Finally we showed that the spinor can be extended to the whole 4-manifold M .

The integral over the mean curvature $H_{T(K)}$ on the RHS of the action can be also expressed by

$$\int_{T(K)} H_{T(K)} \sqrt{g_\partial} d^3 x = \int_{T(K)} \text{tr}(\theta \wedge R)$$

by using a frame θ and the curvature 2-form R . This term can be interpreted as the boundary term of the Einstein-Hilbert action. As shown by York [61], the fixing of the conformal class of the spatial metric in the ADM formalism leads to a boundary term which can be also found in the work of Hawking and Gibbons [62]. Also Ashtekar et.al. [63, 64] discussed the boundary term in the Palatini formalism. Therefore we have

$$\int_{T(K)} \text{tr}(\theta \wedge R) \rightarrow \int_{M(K)} R_M \sqrt{g_M} d^4 x = S_{EH}$$

with $\partial M(K) = T(K)$. Thus we obtain the Einstein-Hilbert action for the target space of the knotted torus $T(K)$ as representation of the immersed disk. Up to now we have studied the immersion of the disk (as well its neighborhood via its boundary the knotted solid torus $T(K)$) and the relation to the target space. One problem is left: the connection of the disks to each other. Geometrically we have a cobordisms between two knotted solid tori $T(K_n)$ and $T(K_m)$, the connecting tube $T(K_n, K_m)$. The corresponding integral

$$\int_{\partial T(K_n, K_m)} H_{T(K_n, K_m)} \sqrt{g} d^3 x = \int_{\partial T(K_n, K_m)} \text{tr}(\theta \wedge F)$$

can be convert to

$$S(\partial T(K_n, K_m)) = \int_{T(K_n, K_m)} \text{tr}(\tilde{F} \wedge \tilde{F}) = \pm \int_{T(K_n, K_m)} \text{tr}(\tilde{F} \wedge * \tilde{F})$$

by using the Stokes theorem, the extension of the curvature F on $\partial T(K_n, K_m)$ to \tilde{F} on $T(K_n, K_m)$ and the instanton equation $\tilde{F} = \pm * \tilde{F}$ along the tube. But then we have the Yang-Mills action for the curvature \tilde{F} . We will not discuss the details and refer to [49]. Finally we will obtain the correct gauge group $U(1) \times SU(2) \times SU(3)$ and get the action

$$S(M) = \int_M \left(R + \sum_n (\bar{\Phi} D^M \Phi)_n \right) \sqrt{g} d^4 x + \sum_{n, m} \int_M \text{tr}(\tilde{F} \wedge * \tilde{F})_{nm}. \quad (19)$$

As conclusion we can state that an immersed disk used in the construction of exotic \mathbb{R}^4 are described by a parallel spinor Φ . The correspondence goes further because the spinor Φ as solution of the Dirac equation (18) is not only generated by a propagator but also by the immersed disk itself. The Feynman path integral of this action can be rearranged by a simply reorganization of the perturbative series in terms of trees [65]. It should be especially emphasized that this method do not need any discretization of the phase space or cluster expansion. Then we obtain a close relation between trees and renormalization similar to approach of Connes and Kreimer [66]. We close this paper with these conjectural remarks.

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